

# VARIANTS OF ENTROPY POWER INEQUALITY \*

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## Abstract

An extension of the entropy power inequality to the form  $N_r^\alpha(X + Y) \geq N_r^\alpha(X) + N_r^\alpha(Y)$  with arbitrary independent summands  $X$  and  $Y$  in  $\mathbb{R}^n$  is obtained for the Rényi entropy and powers  $\alpha \geq (r + 1)/2$ .

## 1 Introduction

Given a continuous random vector  $X$  in  $\mathbb{R}^n$  with density  $f$ , define the (Shannon) entropy and the associated entropy power

$$h(X) = - \int_{\mathbb{R}^n} f(x) \log f(x) dx, \quad N(f) = N(X) = e^{\frac{2}{n} h(X)}.$$

Serving as measures of “chaos” or “randomness” hidden in the distribution of  $X$ , these functionals possess a number of remarkable properties, especially when they are considered on convolutions. For example, we have the famous entropy power inequality (EPI), fundamental in Information Theory, which states that

$$N(X + Y) \geq N(X) + N(Y), \quad (1.1)$$

for arbitrary independent summands  $X$  and  $Y$  in  $\mathbb{R}^n$  (cf. [Sh], [St]). Several proofs of the EPI exist (see e.g. [L], [D-T-C], [S-V], [V-G], [R], [W-M]), as well as refinements (see e.g. [A-B-B-N], [M-B], [Co]). We refer to the survey [M-M-X] for further details. Moreover, when a Gaussian noise is added to  $X$ , i.e., if  $Y = \sqrt{t}Z$  with  $Z$  standard normal, the random vector  $X + \sqrt{t}Z$  has density  $f_t$  whose entropy power is a concave function in  $t$ , so that

$$\frac{d^2}{dt^2} N(f_t) \leq 0 \quad (t > 0). \quad (1.2)$$

This observation due to Costa [C], which strengthens (1.1) in the special case where  $Y$  is Gaussian, is known as the concavity of entropy power theorem (cf. also [D], [V]).

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Of a large interest there has been the question on possible extensions of such properties to more general informational characteristics given, in particular, by the Rényi entropy power

$$N_r(X) = \left( \int_{\mathbb{R}^n} f(x)^r dx \right)^{-\frac{2}{n} \frac{1}{r-1}}$$

of a fixed order  $r > 0$ , or by some natural functionals of  $N_r$ . As one interesting example, for the densities  $u(x, t) = f_t(x)$  solving the nonlinear heat equation  $\frac{\partial}{\partial t} u = \Delta u^r$  with  $r > 1 - \frac{2}{n}$ , Savaré and Toscani [S-T] have extended property (1.2) to the functional  $N_r^\alpha$  in place of  $N$ , where  $\alpha = 1 + \frac{n}{2}(r-1)$ . Therefore, in this PDE context, it is natural to work with

$$\tilde{N}_r(X) = \left( \int_{\mathbb{R}^n} f(x)^r dx \right)^{-\frac{2}{n} \frac{1}{r-1} - 1},$$

called the  $r$ -th Rényi power in [S-T]. Although the solutions  $f_t$  lose the convolution structure, one may wonder whether or not the Savaré-Toscani entropy power  $\tilde{N}_r$  shares the EPI (1.1) as well. Here we give an affirmative answer to this question, including sharper powers of  $N_r$ .

**Theorem 1.1.** *Given independent random vectors  $X$  and  $Y$  in  $\mathbb{R}^n$  with densities, we have*

$$N_r^\alpha(X + Y) \geq N_r^\alpha(X) + N_r^\alpha(Y) \quad (1.3)$$

whenever  $\alpha \geq \frac{r+1}{2}$  ( $r > 1$ ).

Letting  $r \downarrow 1$ , inequality (1.3) returns us to the classical EPI. This inequality is getting sharper when  $r$  is fixed and  $\alpha$  decreases, but it is not clear to us, if the power  $\alpha = \frac{r+1}{2}$  is critical (which appears in the definition of  $\tilde{N}_r$  for dimension one). Anyhow, (1.3) is no longer true for  $\alpha = 1$  like in (1.1). For the range  $r > 3$ , this fact was mentioned in [B-C] in case where both  $X$  and  $Y$  are uniformly distributed. As we will see, (1.3) may be violated with  $\alpha = 1$  for any  $r > 1$ , even when one of the summands is normally distributed (that is, for the densities  $f_t$  in the heat semigroup model).

In the proof of (1.3) we follow an approach of Lieb [L], employing Young's inequality with best constants. Although the basic argument is rather standard, we remind it in the next section. In our situation it leads to some routine calculus computations, so we move the involved analysis to separate sections (starting with the case of equal entropy powers). In Section 5, we analyze (1.3) with  $\alpha = 1$  and show that this inequality cannot be true in general. Finally, in Section 6 we provide a simple lower bound on the optimal exponent  $\alpha = \alpha(r)$  in (1.3).

## 2 Information-theoretic formulation of Young's inequality

The Young inequality with optimal constants (due to Beckner [B1] and Brascamp and Lieb [B-L]) indicates that, for any two independent random vectors  $X$  and  $Y$  in  $\mathbb{R}^n$  with densities  $f$  and  $g$ , respectively, and for all parameters  $p, q, r \geq 1$  such that

$$\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}, \quad (2.1)$$

we have

$$\|f * g\|_r \leq C^{\frac{n}{2}} \|f\|_p \|g\|_q \quad (2.2)$$

with

$$C = C(p, q, r) = \frac{c_p c_q}{c_r}, \quad \text{where } c_\alpha = \frac{\alpha^{1/\alpha}}{(\alpha')^{1/\alpha'}}. \quad (2.3)$$

As usual,  $f * g$  denotes the convolution,  $p' = \frac{p}{p-1}$  is the conjugate power, and

$$\|f\|_p = \left( \int_{\mathbb{R}^n} f(x)^p dx \right)^{1/p}$$

stands for the  $L^p$ -norm of a non-negative function  $f$  on  $\mathbb{R}^n$  with respect to the Lebesgue measure. In general, we have  $C \leq 1$ , with equality in (2.2) attainable for Gaussian densities (The traditional Young inequality is formulated without this constant, so, in a weaker form).

Since  $\|f\|_r = N_r(X)^{-\frac{n}{2r'}}$ , the inequality (2.2) may be stated as a dimension-free relation between the corresponding entropy powers, namely

$$N_r(X + Y)^{\frac{1}{r'}} \geq \frac{1}{C} N_p(X)^{\frac{1}{p'}} N_q(Y)^{\frac{1}{q'}}. \quad (2.4)$$

This is an equivalent information-theoretic formulation of Beckner's result, specialized to the class of probability densities, which appears, for example, in the book by Cover and Thomas [C-T] (in a slightly different form, cf. Theorem 17.8.3, p. 677).

It is natural to have an analog of (2.4) for one functional  $N_r$  only (rather than for three parameters). This can be done on the basis of (2.2) by noting that, due to Jensen's (or Hölder's) inequality, and since  $p, q \leq r$  in (2.1) and  $f$  is a probability density function,

$$\|f\|_p^p \leq \|f\|_1^{1-\frac{p-1}{r-1}} \|f\|_r^{\frac{p-1}{r-1}} = \|f\|_r^{\frac{p-1}{r-1}} \quad (r > 1).$$

As an alternative approach, one can just use the monotonicity of the function  $r \rightarrow N_r$ , which follows, for example, from the representation

$$N_r^{-\frac{n}{2}}(X) = [\mathbb{E} f(X)^{r-1}]^{\frac{1}{r-1}}.$$

Hence  $N_p \geq N_r$ ,  $N_q \geq N_r$  in (2.4), and with these bounds it immediately yields:

**Proposition 2.1.** *Given independent random vectors  $X$  and  $Y$  in  $\mathbb{R}^n$  with densities, we have*

$$N_r(X + Y)^{\frac{1}{r'}} \geq \frac{1}{C} N_r(X)^{\frac{1}{p'}} N_r(Y)^{\frac{1}{q'}}, \quad (2.5)$$

which holds true for all  $p, q, r \geq 1$  subject to (2.1) with constant  $C = C(p, q, r)$  as in (2.3).

A weak point of this inequality is however the loss of equality on Gaussian densities. Nevertheless, there is still freedom to optimize the right-hand side over all admissible couples  $(p, q)$ , or to choose specific values, even if they are not optimal.

### 3 The case of equal entropy powers

Let us illustrate this approach in the simpler situation of equal Rényi entropies. When  $N_r(X) = N_r(Y) = N$ , inequality (2.5) is simplified to

$$N_r(X + Y) \geq C^{-r'} N, \quad (3.1)$$

and our task reduces to the minimization of  $C$  as a function of  $(p, q)$  for a fixed  $r > 1$ . Putting  $x = \frac{1}{p'}$ ,  $y = \frac{1}{q'}$ , so that  $\frac{1}{p} = 1 - \frac{1}{p'} = 1 - x$  and  $\frac{1}{q} = 1 - \frac{1}{q'} = 1 - y$ , from (2.3),

$$\frac{1}{C} = \frac{c_r}{c_p c_q} = c_r \frac{(\frac{1}{p})^{1/p}}{(\frac{1}{p'})^{1/p'}} \frac{(\frac{1}{q})^{1/q}}{(\frac{1}{q'})^{1/q'}} = c_r \frac{(1-x)^{1-x}}{x^x} \frac{(1-y)^{1-y}}{y^y}. \quad (3.2)$$

Hence, we need to maximize the quantity

$$\begin{aligned} \psi(x) &= \log \frac{1}{C} \\ &= \log c_r - (x \log x - (1-x) \log(1-x)) - (y \log y - (1-y) \log(1-y)) \end{aligned}$$

subject to the constraint (2.1), that is, for  $x, y \geq 0$ ,  $x + y = \frac{1}{r'}$ , or equivalently, on the interval  $0 \leq x \leq \frac{1}{r'}$ . At the endpoints, we have  $\psi(0) = \psi(1/r') = 0$ , while inside the interval

$$\psi'(x) = \log \frac{y(1-y)}{x(1-x)} = 0$$

if and only if  $y(1-y) = x(1-x)$ . This equation is solved either as  $y = x = \frac{1}{2r'}$  or as  $y = 1-x$ . But the latter contradicts  $x + y = \frac{1}{r'} < 1$ . Moreover,

$$\psi''(x) = \frac{2x-1}{x(1-x)} + \frac{2y-1}{y(1-y)}$$

is negative at  $x_0 = \frac{1}{2r'}$ , which implies that  $x_0$  is the point of maximum of the function  $\psi$ .

Thus, the coefficient  $\frac{1}{C}$  in (2.5) is maximized, when  $p' = q' = 2r' = \frac{2r}{r-1}$ . For these values,  $p = q = \frac{2r}{r+1}$ , so

$$c_p = c_q = (2r)^{\frac{1}{r}} (r-1)^{\frac{r-1}{2r}} (r+1)^{-\frac{r+1}{2r}} \quad \text{and} \quad C = 2^{\frac{2}{r}} r (r+1)^{-\frac{r+1}{r}}.$$

It remains to raise  $C$  to the power  $-r' = -\frac{r}{r-1}$ , and then we obtain an explicit expression for the optimal constant in (3.1) derived on the basis of (2.5).

**Proposition 3.1.** *If the independent random vectors  $X$  and  $Y$  satisfy  $N_r(X) = N_r(Y) = N$  for some  $r \geq 1$ , then*

$$N_r(X + Y) \geq A_r N, \quad A_r = 4^{-\frac{1}{r-1}} (r+1)^{\frac{r+1}{r-1}} r^{-\frac{r}{r-1}}. \quad (3.3)$$

Proposition 3.1 is not new and a more general version where the distributions have different Rényi entropies was obtained in [R-S, Theorem 1]. Moreover, in the case of different Rényi entropies, tighter bounds are provided in [R-S, Corollary 3].

Now, it is easy to see that  $1 < A_r < 2$ . Moreover, (3.3) provides the desired linear bound (1.3) in case of equal entropy powers,  $N_r^\alpha(X + Y) \geq 2N^\alpha(X)$ , as long as  $A_r \geq 2^{1/\alpha}$ , or equivalently, when  $\alpha > \alpha(r) = (\log 2)/(\log A_r)$ . A simple analysis shows that  $\alpha(r) \leq (r+1)/2$ .

## 4 The general case

Here we derive the extension (1.3) of the EPI for the power  $\alpha = \frac{r+1}{2}$  in the case of arbitrary values  $N_r(X)$  and  $N_r(Y)$ . As a preliminary step, let us return to the inequality (2.5) and raise it to the power  $\alpha$ , so as to rewrite it as

$$N_r^\alpha(X+Y) \geq C^{-\alpha r'} N_r(X)^{\frac{\alpha r'}{p'}} N_r(Y)^{\frac{\alpha r'}{q'}}.$$

Putting  $x = N_r^\alpha(X)$ ,  $y = N_r^\alpha(Y)$  and assuming without loss of generality that  $x + y = \frac{1}{r'}$  (using homogeneity of these functionals), we need to show that

$$C^{-\alpha r'} x^{\frac{r'}{p'}} y^{\frac{r'}{q'}} \geq \frac{1}{r'}$$

for some admissible  $p, q \geq 1$ , i.e., satisfying the condition  $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$ . The best values of  $p$  and  $q$  can be described implicitly as solutions to a certain equation, and we prefer to take some specific values. As a natural choice, consider  $(p, q)$  such that  $\frac{1}{p'} = x$  and  $\frac{1}{q'} = y$  and try to check the desired inequality  $x^{xr'} y^{yr'} \geq C^{\alpha r'} \frac{1}{r'}$ , i.e.,

$$x^x y^y \geq C^\alpha \frac{1}{(r')^{1/r'}} \quad \left( x, y > 0, x + y = \frac{1}{r'} \right).$$

Equivalently, so that to eliminate the parameter  $r$ , we need to check whether or not

$$x^x y^y \geq C^\alpha (x + y)^{x+y} \quad (x, y > 0, x + y < 1). \quad (4.1)$$

As in Section 3, cf. (3.2),

$$C = \frac{c_p c_q}{c_r} = \frac{x^x}{(1-x)^{1-x}} \frac{y^y}{(1-y)^{1-y}} \frac{(1-x-y)^{1-x-y}}{(x+y)^{x+y}},$$

and (4.1) takes the form

$$\left( \frac{(x+y)^{x+y}}{x^x y^y} \right)^{\alpha-1} \geq \left( \frac{(1-x-y)^{1-x-y}}{(1-x)^{1-x} (1-y)^{1-y}} \right)^\alpha. \quad (4.2)$$

Now, in case  $\alpha = \frac{r+1}{2}$ , we have

$$\alpha = \frac{2 - \frac{1}{r'}}{2(1 - \frac{1}{r'})} = \frac{2 - x - y}{2(1 - x - y)}, \quad \alpha - 1 = \frac{x + y}{2(1 - x - y)},$$

and (4.2) becomes

$$\left( \frac{(x+y)^{x+y}}{x^x y^y} \right)^{x+y} \geq \left( \frac{(1-x-y)^{1-x-y}}{(1-x)^{1-x} (1-y)^{1-y}} \right)^{2-x-y}, \quad (4.3)$$

which is to be verified in the triangle  $x, y > 0, x + y < 1$ . This inequality is obviously true for boundary values  $x = 0, y = 0$ , and when  $x + y = 1$ , with equality of both sides. To proceed, we choose an argument based on the following calculus lemma.

**Lemma 4.1.** Given  $0 < c < 1$  and  $\beta \geq \frac{2}{c} - 1$ , the function

$$\psi(x) = \frac{(1-x)^{\beta(1-x)} (1-y)^{\beta(1-y)}}{x^x y^y} \quad (0 \leq x \leq c, \quad y = c - x)$$

attains minimum either at the endpoints  $x = 0$ ,  $x = c$ , or at the center  $x = \frac{c}{2}$ . Moreover, in case  $\beta = \frac{2}{c} - 1$ , this function attains minimum at the endpoints.

**Proof.** Inside the interval  $(0, c)$  the function

$$\begin{aligned} v(x) &= \log \psi(x) \\ &= \beta((1-x) \log(1-x) + (1-y) \log(1-y)) - (x \log x + y \log y) \end{aligned}$$

has the first two derivatives

$$\begin{aligned} v'(x) &= -\beta (\log(1-x) - \log(1-y)) - (\log x - \log y), \\ v''(x) &= \left( \frac{\beta}{1-x} + \frac{\beta}{1-y} \right) - \left( \frac{1}{x} + \frac{1}{y} \right) = \frac{\beta(2-c)}{(1-x)(1-y)} - \frac{c}{xy}. \end{aligned}$$

Note that  $v(0) = v(c)$ . Also,  $v'(0+) = \infty$ ,  $v'(c-) = -\infty$ , so  $v$  is increasing near zero and is decreasing near the point  $c$ . In addition,  $v''(x)$  is vanishing, if and only if

$$w(x) \equiv \beta(2-c)xy - c(1-x)(1-y) = 0, \quad (4.4)$$

which is a quadratic equation (recall that  $y = c - x$ ). In general it has at most two roots.

Case 1: Equation (4.4) has at most one root in  $(0, c)$ . Since  $w(0) < 0$ , it means that  $w(x) \leq 0$  in  $(0, c)$ . Therefore,  $v$  is concave, and thus attains its minimum at the endpoints of this interval.

Case 2: Equation (4.4) has exactly two roots in  $(0, c)$ , say  $0 < x_1 < x_2 < c$ . Since  $w(0) < 0$  and  $w(c) < 0$ , it means that  $w(x) < 0$  in  $(0, x_1)$  and  $(x_2, c)$ , while  $w(x) > 0$  in  $(x_1, x_2)$ . That is,  $v$  is strictly concave on  $(0, x_1)$  and  $(x_2, c)$ , and is strictly convex on the intermediate interval. Hence, in this case there is at most one point of local minimum. If there is no point of local minimum, then  $v$  attains its minimum at the endpoints. If there is one point  $x_0$  of local minimum of  $v$ , then it must belong to  $(x_1, x_2)$ , and there are two points of local maximum, say  $z_1$  and  $z_2$  belonging to the other subintervals. In particular,  $v' < 0$  on  $(z_1, x_0)$  and  $v' > 0$  on  $(x_0, z_2)$ .

Note that  $v'(c/2) = 0$ , so this point is a candidate for local extremum. Moreover, by the assumption on  $\beta$ ,

$$v''(c/2) = \frac{4(\beta c - (2-c))}{c(2-c)} \geq 0.$$

If  $\beta > \frac{2}{c} - 1$ , then  $v''(c/2) > 0$  which means that  $x_0 = c/2$  is a local minimum for  $v$  and therefore for  $\psi$ , and the first assertion follows. If  $\beta = \frac{2}{c} - 1$ , then  $v''(c/2) = 0$  which means that either  $c/2 = x_1$  or  $c/2 = x_2$ . But at these points the derivative of  $v$  may not vanish. In other words, the equality  $\beta = \frac{2}{c} - 1$  is only possible under Case 1.  $\square$

**Proof of Theorem 1.1.** Let us return to the inequality (4.3) and rewrite it as

$$\frac{(x+y)^{x+y}}{x^x y^y} \geq \left( \frac{(1-x-y)^{1-x-y}}{(1-x)^{1-x} (1-y)^{1-y}} \right)^\beta, \quad \beta = \frac{\alpha}{\alpha-1},$$

or equivalently

$$\frac{(1-x)^{\beta(1-x)} (1-y)^{\beta(1-y)}}{x^x y^y} \geq \frac{(1-x-y)^{\beta(1-x-y)}}{(x+y)^{x+y}}. \quad (4.5)$$

Here the right-hand side depends only on  $c = x+y$  (since  $\alpha$  may only depend on  $r$  which is a function of  $x+y$ ). Hence, to prove (4.5), it is sufficient to minimize the left-hand side under the constraint  $x, y \geq 0$ ,  $x+y = c$ , and then to compare the minimum with the right-hand side. In case  $\alpha = \frac{r+1}{2}$ , we have

$$\frac{\alpha}{\alpha-1} = \frac{2-x-y}{x+y} = \frac{2-c}{c} = \frac{2}{c} - 1,$$

which is exactly the extreme value for  $\beta$  in Lemma 4.1. Therefore, by its conclusion, the left-hand side of (4.5) is minimized either at  $x = 0$  or  $x = c$ . But for such boundary values there is equality in (4.5), as was already stressed before Lemma 4.1. As a result, we obtain the desired inequality (4.3) for all  $x, y \geq 0$  such that  $x+y \leq 1$ .  $\square$

## 5 Rényi entropy powers for the heat semi-group

Let us now look at the possible behavior of the Rényi entropy powers in the class of densities  $f_t$  of  $X_t = X + \sqrt{t}Z$ , assuming that  $X$  has a sufficiently regular positive density  $f$  (on the line), and  $Z$  is a standard normal random variable independent of  $X$ . Since for small  $t > 0$

$$f_t(x) = f(x) + \frac{1}{2} f''(x) t + o(t),$$

we find, by Taylor expansion and integrating by parts,

$$\int_{-\infty}^{\infty} f_t(x)^r dx = \int_{-\infty}^{\infty} f(x)^r dx - \frac{t}{2} r(r-1) \int_{-\infty}^{\infty} f(x)^{r-2} f'(x)^2 dx + o(t),$$

and thus, for  $r > 1$ ,

$$N_r(X_t) = N_r(X) + tr \left( \int_{-\infty}^{\infty} f(x)^r dx \right)^{\frac{1+r}{1-r}} \int_{-\infty}^{\infty} f(x)^{r-2} f'(x)^2 dx + o(t).$$

Using this representation, we are going to test the inequality (1.3) for  $\alpha = 1$ , when it becomes

$$N_r(X_t) \geq N_r(X) + t N_r(Z).$$

Comparing the linear terms in front of  $t$  and using  $N_r(Z) = 2\pi r^{\frac{1}{r-1}}$ , we would be led to a Nash-type inequality

$$r \left( \int_{-\infty}^{\infty} f(x)^r dx \right)^{\frac{1+r}{1-r}} \int_{-\infty}^{\infty} f(x)^{r-2} f'(x)^2 dx \geq 2\pi r^{\frac{1}{r-1}}, \quad (5.1)$$

holding already without too restrictive conditions (e.g., for all  $C^1$ -smooth  $f > 0$ ).

Now, let us take  $f(x) = B e^{-\frac{|x|^p}{p}}$  with  $p \geq 2$ , where  $B$  is a normalizing constant, i.e.,  $B^{-1} = 2 p^{\frac{1}{p}-1} \Gamma(\frac{1}{p})$ . In this case,

$$\int_{-\infty}^{\infty} f(x)^r dx = B^r \int_{-\infty}^{\infty} e^{-\frac{r|x|^p}{p}} dx = B^{r-1} \frac{1}{r^{1/p}},$$

so that

$$\left( \int_{-\infty}^{\infty} f(x)^r dx \right)^{\frac{1+r}{1-r}} = B^{-(r+1)} r^{\frac{r+1}{p(r-1)}}.$$

Similarly,

$$\int_{-\infty}^{\infty} f'(x)^2 f(x)^{r-2} dx = B^r \int_{-\infty}^{\infty} |x|^{2(p-1)} e^{-\frac{r|x|^p}{p}} dx = 2B^r \left(\frac{p}{r}\right)^{\frac{2p-1}{p}} \frac{1}{p} \Gamma\left(2 - \frac{1}{p}\right),$$

and thus the left-hand side in (5.1) is equal to

$$\frac{2r}{B} r^{\frac{r+1}{p(r-1)}} \left(\frac{p}{r}\right)^{\frac{2p-1}{p}} \frac{1}{p} \Gamma\left(2 - \frac{1}{p}\right) = 4 \Gamma\left(\frac{1}{p}\right) \Gamma\left(2 - \frac{1}{p}\right) r^{\frac{2r}{p(r-1)}-1}.$$

Hence, inequality (5.1) says that

$$2\pi \leq 4 r^{-\frac{r(p-2)}{p(r-1)}} \Gamma\left(\frac{1}{p}\right) \Gamma\left(2 - \frac{1}{p}\right). \quad (5.2)$$

We claim that it cannot be true for all  $p > 2$  sufficiently close to 2 (i.e., when  $X$  itself is almost standard normal). To see this, denote by  $G(p)$  the right-hand side of (5.2) and note that there is equality at  $p = 2$ . So, let us look at the derivative and show that  $G'(2) < 0$ , i.e.,  $H'(1/2) > 0$  for  $H(x) = \log G(1/x)$ . Indeed,

$$H'(x) = \frac{2r}{r-1} \log r + \frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(2-x)}{\Gamma(2-x)}.$$

From the fundamental relation  $\Gamma(x+1) = x\Gamma(x)$ , it follows that  $\Gamma'(x+1) = \Gamma(x) + x\Gamma'(x)$ , so  $\Gamma'(3/2) = \Gamma(1/2) + \frac{1}{2}\Gamma'(1/2)$ , while  $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2)$ . Hence,

$$H'(1/2) = \frac{2r}{r-1} \log r - 2 > 0.$$

We may conclude that the entropy power inequality for  $N_r$  of any order  $r > 1$  does not hold in general, even when one of the variable is Gaussian.

For another, less direct argument, one may return to (5.1) and rewrite it as a homogeneous inequality

$$r \left( \int_{\mathbb{R}} f(x)^r dx \right)^{\frac{1+r}{1-r}} \int_{\mathbb{R}} f(x)^{r-2} f'(x)^2 dx \geq 2\pi r^{\frac{1}{r-1}} \left( \int_{\mathbb{R}} f(x) dx \right)^{\frac{2r}{1-r}}.$$

After the change  $f = u^{\frac{2}{r}}$ , it takes the form of the Nash-type inequality

$$\left( \int_{\mathbb{R}} u(x)^2 dx \right)^{\frac{r+1}{r-1}} \leq K_r \int_{\mathbb{R}} u'(x)^2 dx \left( \int_{\mathbb{R}} u(x)^{\frac{2}{r}} dx \right)^{\frac{2r}{r-1}} \quad (5.3)$$



with  $K_r = \frac{2}{\pi r^{r-1}}$ . In fact, the Nash inequality in  $\mathbb{R}^n$  asserts that

$$\left( \int_{\mathbb{R}^n} u(x)^2 dx \right)^{1+\frac{2}{n}} \leq C_n \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \left( \int_{\mathbb{R}^n} u(x) dx \right)^{\frac{4}{n}}$$

with sharp constant given by

$$C_n = \left(1 + \frac{2}{n}\right) \Gamma\left(\frac{n}{2} + 2\right)^{\frac{2}{n}} \frac{1}{\pi j_{n/2}^2}$$

(cf. [C-L], [B2]). Here  $j_{n/2}$  denotes the smallest positive zero of the Bessel function  $J_{n/2}$  of order  $n/2$ . In dimension  $n = 1$ , one has  $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$  (cf. [W], p.54, eq. (3)), thus  $j_{1/2} = \pi$ . Hence the sharp Nash inequality in dimension 1 reads

$$\left( \int u(x)^2 dx \right)^3 \leq \frac{27}{16\pi^2} \int u'(x)^2 dx \left( \int u(x) dx \right)^4,$$

which is the same as (5.3) for  $r = 2$ , however, with a larger constant. Hence, as we have already seen, inequality (1.3) cannot be true for  $\alpha = 1$  and  $r = 2$ . Let us notice that the Nash inequality with the asymptotically sharp constant  $2/(\pi en)$  can be deduced from the classical EPI (1.1) (cf. [T]).

For the parameter  $r = 2$ , routine computations also provide a counter-example in the case where both  $X$  and  $Y$  have the beta distribution with density  $f(x) = \frac{3}{4}(1 - x^2)$ ,  $|x| < 1$ .

## 6 Lower bound on the optimal exponent

One may also provide a simple lower bound on the optimal exponent  $\alpha = \alpha_{opt}$  that satisfies the inequality

$$N_r(X + Y)^\alpha \geq N_r(X)^\alpha + N_r(Y)^\alpha$$

for all independent random vectors  $X$  and  $Y$ . Together with the upper bound of Theorem 1.1, we have:

**Proposition 6.1** *One has  $\alpha_{opt} \in \left[ \frac{\log 2}{2} \frac{r-1}{\log(\frac{r+1}{2})}, \frac{r+1}{2} \right]$ .*

**Proof.** For the lower bound, let  $X$  and  $Y$  be independent and uniformly distributed on  $[0, 1]$ , in which case  $N_r(X) = N_r(Y) = 1$ . The sum  $X + Y$  has the triangle density  $(f * g)(x) = x$  on  $[0, 1]$  and  $(f * g)(x) = 2 - x$  on  $[1, 2]$ . Hence,

$$\int (f * g)(x)^r dx = \int_0^1 x^r dx + \int_1^2 (2 - x)^r dx = \frac{2}{r+1}.$$

Thus

$$N_r(X + Y) = \left( \frac{r+1}{2} \right)^{\frac{2}{r-1}}.$$

Since  $N_r(X + Y)^{\alpha_{opt}} \geq N_r(X)^{\alpha_{opt}} + N_r(Y)^{\alpha_{opt}}$ , we deduce that  $\left(\frac{r+1}{2}\right)^{\frac{2\alpha_{opt}}{r-1}} \geq 2$ , which is the required statement.

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